Sheaves on ALE spaces and quiver varieties

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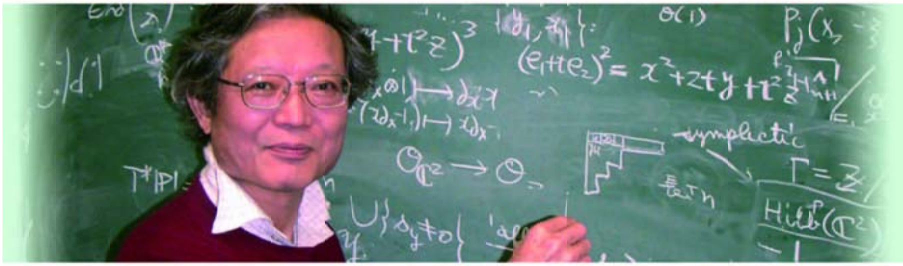


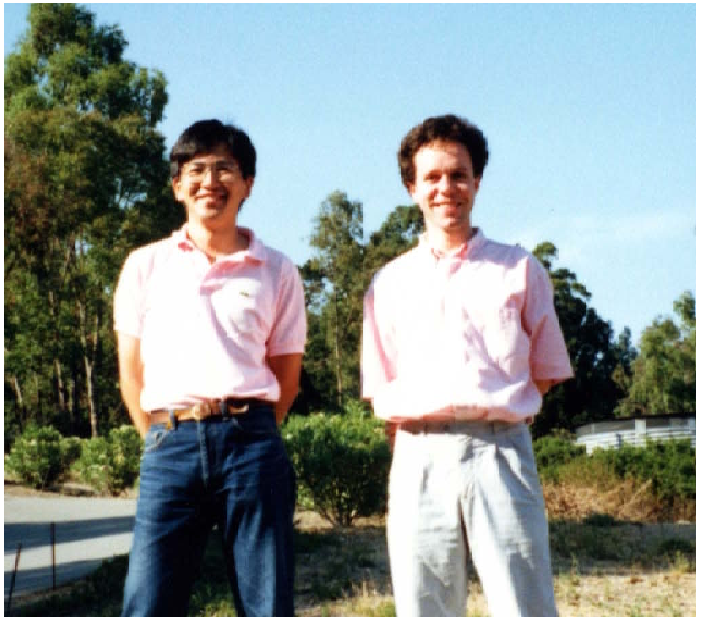
Algebraic Analysis and Around
in honor of Professor Masaki Kashiwara's 60th birthday

June 25 - 30, 2007 Kyoto

1989 Summer, MSRI Kronheimer - N

A description of Yang-Mills instantons on an ALE space
(generalization of Atiyah-Drinfeld-Hitchin-Manin)

In particular,
their moduli space = moduli space of representations
of a quiver



I didn't have contact with Professor Kashiwara
at this moment.

1992〜93

I defined quiver varieties as
generalization of the moduli spaces.

I've got a letter from Kashiwara:

> 前略
> この間は面白い話を有難うございました。
> これについていくつか気のついた事を書きます。

But the stability conditions (parameter used to define the "quotient") are different for instanton moduli spaces and quiver varieties.

Today I want to

① study the change of quiver varieties under move of the stability conditions (stratified Mukai flop, "Jordan" flop),
?

② give the stability condition for torsion-free sheaves.

Today I also want to

3 explain an application to the representation theory of affine Lie alg. (my student Nagao)

But I am sure that I will not have time.

$\longrightarrow$ see math/0703107

4 explain a relation to the representation theory of the rational DAHA, suggested by I. Gordon.

math/0703150

§1.ALE space (constructed by Kronheimer)

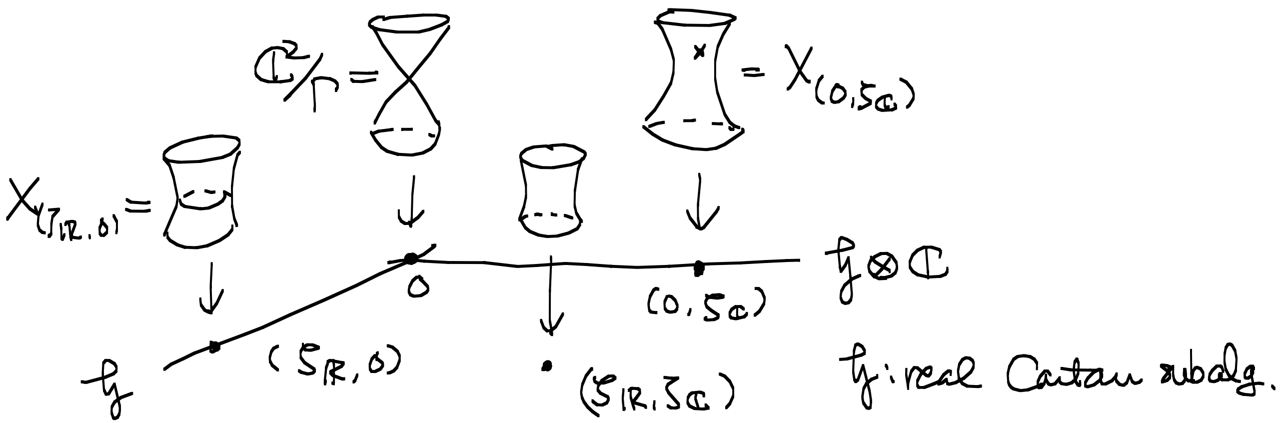
$\Gamma \subset SL_2(\mathbb{C})$ finite subgroup

$\rightsquigarrow$ ADE classification

$\longrightarrow$ simple Lie algebra

$0 \in \mathbb{C}^2/\Gamma$ simple singularity

$\Longrightarrow$ Kronheimer constructed
semiuniversal deformation
and its simultaneous resolution
as moduli spaces of representations of affine quivers,
together with hyperKähler metrics



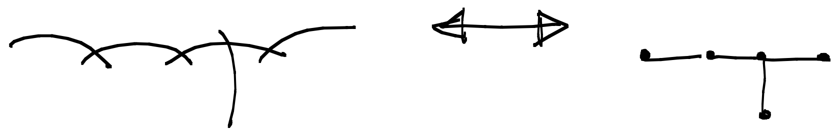
If $\zeta_{\mathbb{C}} = 0$ and $\zeta_{\mathbb{R}}$: "generic" (explained later)

$$\Longrightarrow \quad \pi : X_\zeta = \widetilde{\mathbb{C}^2/\Gamma} \longrightarrow X_0 = \mathbb{C}^2/\Gamma$$

minimal resolution of singularities

$$\pi^{-1}(0) = \bigcup C_i \qquad C_i \cong \mathbb{P}^1$$

Dynkin diagram



framed moduli space of instantons / torsion-free sheaves

$=$ isomorphism classes of

- A : anti-self-dual connection on $\mathcal{E}$ (C^∞ vector b'dle with hermitian inner product)
- E : torsion-free sheaf

on the compactification
- $\hat{X}_\zeta = X_\zeta \cup \text{point}/\Gamma$
- $\overline{X}_\zeta = X_\zeta \cup \text{line}/\Gamma$

together with trivialization (framing)
- $\mathcal{E}|_{pt} \cong \mathcal{E}_0$
- $E|_{line} \cong E_0$

- $\mathcal{E}_0 = W/\Gamma$: orbifold bundle over pt/Γ
- $E_0 = (\mathcal{O}_\ell \otimes_{\mathbb{C}} W)/\Gamma$: orbifold sheaf over line $/\Gamma$

where W is a Γ-module

§2.Quiver varieties

(I, E) : affine Dynkin diagram

$I = \{\text{vertexes}\}$, $E = \{\text{edges}\}$

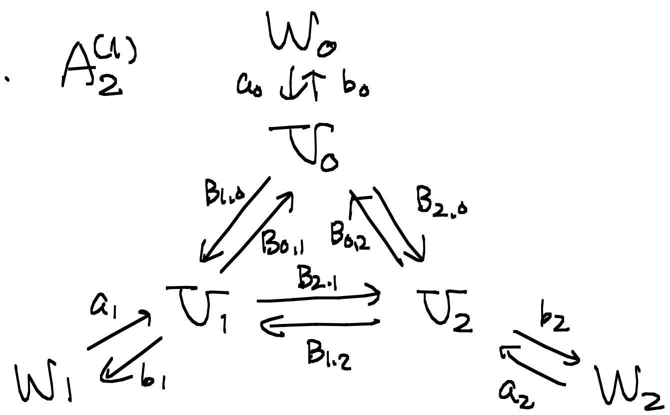
$H = E \sqcup E = \{\text{oriented edges}\}$

V, W : I-graded vector spaces

We identify them with $(\dim V_i), (\dim W_i) \in \mathbb{Z}_{\geq 0}^{I}$.

$$\mathbf{M}(V, W) = \bigoplus_{h \in H} \operatorname{Hom}(V_{o(h)}, V_{i(h)}) \ni B_h \oplus \bigoplus_{i \in I} \underset{\substack{\rotatebox{90}{$\in$} \\ a_i}}{\operatorname{Hom}(W_i, V_i)} \oplus \underset{\substack{\rotatebox{90}{$\in$} \\ b_i}}{\operatorname{Hom}(V_i, W_i)}$$

e.g. $A_2^{(1)}$



group action

$$\mathbb{M}(V, W) \circlearrowleft G_V = \prod_{i \in I} GL(V_i)$$

moment map

$$\mu : \mathbb{M}(V, W) \longrightarrow \mathrm{Lie}\, G_V$$

$$(B_h, a_i, b_i) \longmapsto \sum_{i(h)=i} \varepsilon(h) B_h B_{\bar{h}} + a_i b_i$$

where $\varepsilon(h) = \pm 1$ $\qquad {}_{\varepsilon} H = E_1 \sqcup E_{-1}$

- complex parameter
$\zeta_{\mathbb{C}} = (\zeta_{\mathbb{C}}^{(i)})_{i\in I} \in \mathbb{C}^I$
$\mu_{\mathbb{C}}^{-1}(\oplus \zeta_{\mathbb{C}}^{(i)}\, \mathrm{id}_{V_i})$: level set $\curvearrowleft G_V$

- real parameter (stability condition)
$\zeta_{\mathbb{R}} = (\zeta_{\mathbb{R}}^{(i)}) \in \mathbb{R}^I$

<u>Def.</u> $x = (B_h, a_i, b_i)$ is $\zeta_{\mathbb{R}}$- semi stable

$\Longleftrightarrow$ (1) $^\forall S = \oplus S_i$ I-graded subspace of V

s.t. $B_h(S_{o(h)}) \subset S_{i(h)}$, $S_i \subset \operatorname{Ker} b_i$

$\Longrightarrow$ $\sum_i \zeta_{\mathbb{R}}^{(i)} \dim S_i \leqq 0$

$<$ unless $S = 0$

and (2) $^\forall T = \oplus T_i$

s.t. $B_h(T_{o(h)}) \subset T_{i(h)}$, $\operatorname{Im} a_i \subset T_i$

$\Longrightarrow$ $\sum_i \zeta_{\mathbb{R}}^{(i)} \dim T_i \leqq \sum \zeta_{\mathbb{R}}^{(i)} \dim V_i$

$<$ unless $T = V$

Write

$$\zeta = (\zeta_{\mathbb{R}}, \zeta_{\mathbb{C}}) = \mathbb{R}^I \oplus \mathbb{C}^I = (\mathrm{Im}\,\mathbb{H})^I$$

We define quiver varieties as quotients:

$$\mathfrak{M}_\zeta(V, W) \overset{\text{def.}}{=} \mu^{-1}(\zeta_{\mathbb{C}})^{\zeta_{\mathbb{R}}\text{-semistable}} /\!/ G_V$$

$$\cup \text{ open}$$

$$\mathfrak{M}^s_\zeta(V, W) \overset{\text{def.}}{=} \mu^{-1}(\zeta_{\mathbb{C}})^{\zeta_{\mathbb{R}}\text{-stable}} / G_V$$

$\mathfrak{M}^s_\zeta(V,W)$: nonsingular of dim. $= \sum_i 2 \dim V_i \dim W_i - \sum_{i,j} C_{ij} \dim V_i \dim V_j$

$\mathfrak{M}_\zeta(V,W) \setminus \mathfrak{M}^s_\zeta(V,W)$: singularities

Thus it is natural to ask when $\mathfrak{M}_\zeta \setminus \mathfrak{M}^s_\zeta \neq \phi$, i.e., when $\exists$ $\zeta_{\mathbb{R}}$-semistable, non $\zeta_{\mathbb{R}}$-stable point ?

<u>Prop.</u> If $\zeta \notin (\mathbb{R} \oplus \mathbb{C}) \otimes D_\theta$ for any root hyperplane $D_\theta \subset \mathfrak{h}_{\mathbb{R}}$ with $\theta = \sum \theta_i \alpha_i$ $\quad \theta_i \leqq \dim V_i$

then $\mathfrak{M}_\zeta(V,W) = \mathfrak{M}^s_\zeta(V,W)$.

§3.Wall-crossing

Fix $\zeta_{\mathbb{C}}$, and move $\zeta_{\mathbb{R}}$.

$R(\zeta_{\mathbb{C}}, V)$ (possibly empty)

$\overset{\text{def}}{=} \{ \theta = \sum \theta_i \alpha_i : \text{positive root} \mid \theta_i \leq \dim V_i, \ \langle \theta, \zeta_{\mathbb{C}} \rangle = 0 \}$

Then $\mathbb{R}^I$ has the chamber structure as

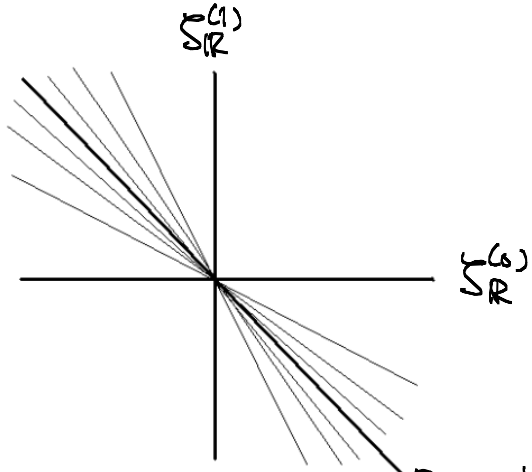
$$\mathbb{R}^I \setminus \bigcup_{\theta \in R(\zeta_{\mathbb{C}}, V)} D_\theta$$

If $\zeta_{\mathbb{R}}$ stays in the connected component,
then $\mathfrak{M}_\zeta$ is unchanged.

<u>Example</u> $A_1^{(1)}$, $\zeta_{\mathbb{C}} = 0$

real roots $= \{ n\alpha_0 + (n+1)\alpha_1,\ (n+1)\alpha_0 + n\alpha_1 \mid n \in \mathbb{Z} \}$

imaginary roots $= \{ n\delta \mid n \in \mathbb{Z} \setminus 0 \}$

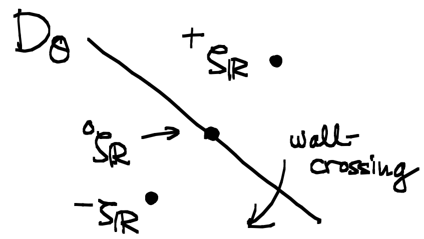


$D_\delta = \{ \zeta_{\mathbb{R}}^{(0)} + \zeta_{\mathbb{R}}^{(1)} = 0 \}$

(level 0 hyperplane)

Fix ∇ $\rightsquigarrow$ we choose finitely many positive roots

$R(\zeta_{\mathbb{C}}, \nabla)$



typical picture in GIT quotients

Suppose $\zeta_{\mathbb{R}}$ cross the wall D_θ.

- (B,a,b) : ${}^\pm\zeta$-stable $\Longrightarrow$ ${}^0\zeta$-semistable.

  $\Longrightarrow$ $\exists$ Jordan-Hölder type filtration by ${}^0\zeta_{\mathbb{R}}$-stable representations

  i.e. $\exists\ V = V^0 \supset V^1 \supset \cdots \supset V^N = 0$

  filtration invariant under (B,a,b) & $\mathrm{gr}_{V^i/V^{i+1}}(B,a,b)$: ${}^0\zeta_{\mathbb{R}}$-stable

  $\Longrightarrow$ Taking direct sum of $\mathrm{gr}_{V^i/V^{i+1}}$,

  we have morphisms $\mathfrak{M}_{+\zeta}, \mathfrak{M}_{-\zeta} \longrightarrow \mathfrak{M}_{0\zeta}$.

- Let ${}^+\mathcal{S}$: ${}^+\zeta_{\mathbb{R}}$-stable, but not ${}^-\zeta_{\mathbb{R}}$-stable,

  ${}^-\mathcal{S}$: ${}^-\zeta_{\mathbb{R}}$-stable, but not ${}^+\zeta_{\mathbb{R}}$-stable.

  $\Longrightarrow$ the images of ${}^+\mathcal{S}$, ${}^-\mathcal{S}$ are in $\mathfrak{M}_{0\zeta} \setminus \mathfrak{M}_{0\zeta}^s$

$$\begin{tikzcd}
\mathfrak{M}_{+\zeta} \supset {}^+\mathcal{S} \arrow[dr] \arrow[ddr] & & {}^-\mathcal{S} \subset \mathfrak{M}_{-\zeta} \arrow[dl] \arrow[ddl] \\
& \mathfrak{M}_{0\zeta} \setminus \mathfrak{M}_{0\zeta}^s \arrow[d, phantom, "\cap"] & \\
& \mathfrak{M}_{0\zeta} &
\end{tikzcd}$$

Next we need to understand ${}^0\zeta_{\mathbb{R}}$-stable representations

$\longrightarrow$ The picture is different for a real root
and an imaginary root

Case 1°. real root $\theta = \sum \theta_i \alpha_i$

a ${}^0\zeta_{\mathbb{R}}$-stable representation is either

a) $\mathfrak{M}^s_{0\zeta}(V', W)$ $W \neq 0$

or b) the unique point B_θ in $\mathfrak{M}_{0\zeta}(\theta, 0)$ (i.e. $\dim V_i = \theta_i$)

(e.g. $A_1^{(1)}$ $\theta = n\alpha_0 + (n+1)\alpha_1$

$$\mathbb{C}^n \underset{\left[\begin{smallmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ 0 & & 0 & 1 \end{smallmatrix}\right]}{\overset{\left[\begin{smallmatrix} 1 & 0 & & 0 \\ & \ddots & \ddots & \\ 0 & & 1 & 0 \end{smallmatrix}\right]}{\leftleftarrows}} \mathbb{C}^{n+1} \quad \text{or} \quad \mathbb{C}^n \underset{\left[\begin{smallmatrix} 1 & & 0 \\ & \ddots & \\ & & 1 \end{smallmatrix}\right]}{\overset{\left[\begin{smallmatrix} 1 & & \\ & \ddots & \\ 0 & & 1 \end{smallmatrix}\right]}{\rightrightarrows}} \mathbb{C}^{n+1}$$
)

$$\therefore \quad \mathfrak{M}_{0\zeta}(V, W) \setminus \mathfrak{M}^s_{0\zeta}(V, W) = \bigsqcup_{\substack{m > 0 \\ V' + m\theta = V}} \mathfrak{M}^s_{0\zeta}(V', W) \times \{[B_\theta^{\oplus m}]\}$$

(stratification)

Moreover the latter has no self-extension.

$\Rightarrow$

- $(B, a, b) \in \mathcal{S}$ is an extension of the form:

$$0 \to \underset{\substack{\in \\ \mathfrak{M}^s_{0_{\zeta_{\mathbb{R}}}}(V', W)}}{(B', a', b')} \to (B, a, b) \longrightarrow B_\theta^{\oplus m} \to 0$$

- $(B, a, b) \in \mathcal{S}^-$ is an extension of the form:

$$0 \to B_\theta^{\oplus m} \longrightarrow (B, a, b) \longrightarrow (B', a', b') \longrightarrow 0$$

$\Rightarrow$ Over each stratum (i.e. fixed $m \in \mathbb{Z}_{>0}$), we have a Grassmann bundle.

$$\Rightarrow \mathcal{S}^+ \cong \bigcup_{\substack{m>0 \\ V = V' \oplus m\theta}} \bigcup_{\substack{(B', a', b') \\ \in \mathfrak{M}^s_{0_{\zeta_{\mathbb{R}}}}(V', W)}} Gr(m, \mathrm{Ext}^1(B_\theta, (B', a', b')))$$

$$\mathcal{S}^- \cong \bigcup_{m} \bigcup_{(B', a', b')} Gr(m, \mathrm{Ext}^1((B', a', b'), B_\theta))$$

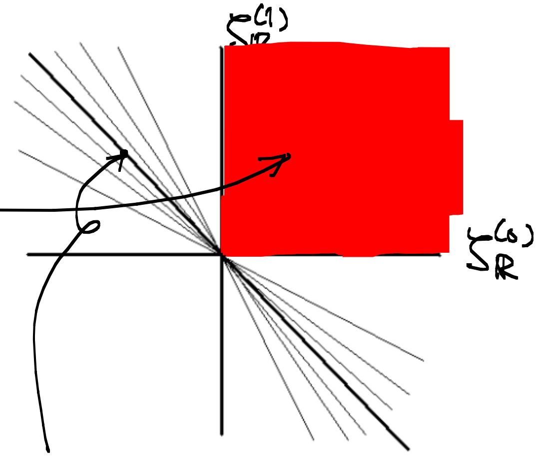
Thus $\mathfrak{M}_{+\zeta}$ and $\mathfrak{M}_{-\zeta}$ are related by a stratified Mukai flop.

Case 2° : $\theta = \delta$ $\rightsquigarrow$ discussed later

§4.Torsion−free sheaves on an ALE space

The usual stability parameter, which I have been used to construct the Kac-Moody affine Lie algebra action on homology groups, is

$\zeta_{\mathbb{R}}^{(i)} > 0 \quad \forall i$



The ALE space X_ζ is an example of quiver varieties with

$$\begin{cases} V = \delta \text{ (imaginary root)} \\ W = 0 \\ \zeta : \text{ in the level } 0 \text{ hyperplane} \end{cases}$$

If $\zeta \notin (\mathbb{R}\oplus\mathbb{C})\otimes D_\theta$ for $\forall$real root θ, X_ζ : nonsingular

<u>Rem</u>. When $W=0$, $\mathbb{C}^* (\subset G_V = \prod GL(V_i))$ acts trivially on $\mathfrak{M}(V,W)$. So we should consider $G_V/\mathbb{C}^*$ instead of G_V.

In particular, $\dim. = 2 - \sum C_{ij} \dim V_i \dim V_j$
$= 2$ if $V = \delta$

Th (Kronheimer + N.)

ζ : as above

$\mathfrak{M}^s_\zeta(V,W)$ = framed moduli space of Yang-Mills instantons on the ALE space $X_{-\zeta}$.

Later I found that if $\zeta_{\mathbb{C}} = 0$ and $\zeta^{(i)}_{\mathbb{R}} < 0 \quad \forall i$

$\Longrightarrow \quad \mathfrak{M}_\zeta(V,W) \cong$ framed moduli space of Γ-equivariant torsion-free sheaves on $\mathbb{C}^2$.

Rem. categorical McKay correspondence

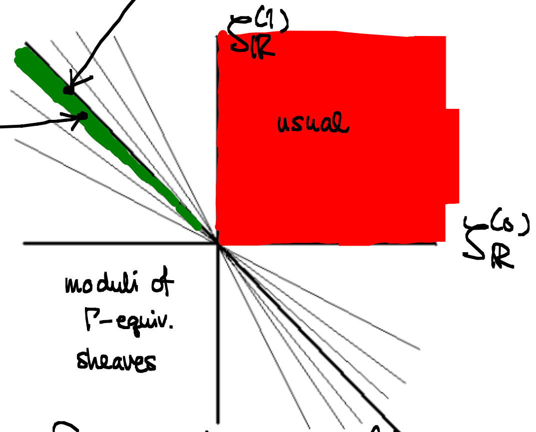
Gonzalez-Sprinberg + Verdier, Kapranov-Vasserot

$$D^b(\Gamma\text{-Coh}\,\mathbb{C}^2) \cong D^b(\mathrm{Coh}\,\widetilde{\mathbb{C}^2/\Gamma})$$

Main Theorem

$\overset{\circ}{\zeta}$: parameter for ALE space as above, i.e.
level 0 hyperplane s.t. $\overset{\circ}{\zeta} \notin (\mathbb{R}\oplus\mathbb{C})\otimes D_\theta$ for $\forall$ real root θ

ζ : from the adjacent chamber



$\Longrightarrow \mathfrak{M}_\zeta(V,W) =$ framed moduli space of torsion-free sheaves on $X_{-\zeta}$

NB. $\zeta_{\mathbb{C}}=0$, $W=\mathbb{C}$ at vertex 0, $V=n\delta$

$\Longrightarrow \mathfrak{M}_\zeta(V,W)$ Hilbert scheme of n points on $\mathbb{C}^2/\Gamma$

This special case was proved by Kuznetsov.

About proof:

straightforward combination of two techiniques

a) Kronheimer + N

b) special case $\Gamma = \{e\}$ torsion-free sheaf on $\mathbb{C}^2$

(Barth, reproduced in my lecture note.)

More words on the proof:

We need

1) a resolution of the diagonal $\Delta \subset X_\zeta \times X_\zeta$.

2) vanishing thm $H^i(\overline{X}_\zeta, \mathcal{R}^* \otimes \mathcal{R})$.

Both were proved in Kronheimer + N.

§5. Wall-crossing revisited

Yoshioka, in private communication, tells me that other $\mathfrak{M}_\zeta(V,W)$'s are also framed moduli spaces of sheaves on $X_{-\zeta}$, not necessarily torsion-free, if $\zeta_{\mathbb{R}} \cdot \delta < 0$.

Then the exact sequence

$$0 \longrightarrow (B', a', b') \longrightarrow (B, a, b) \longrightarrow B_\theta^{\oplus m} \longrightarrow 0$$

can be identified with the exact sequence in $\mathrm{Coh}\, X_{-\zeta}$.

$B_\theta \leftrightarrow$ a torsion sheaf supported on curves

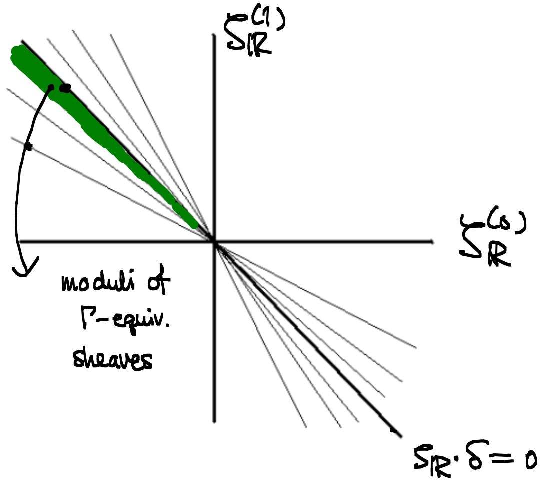
After the wall-crossing,

$$0 \to B_\theta^{\oplus m} \longrightarrow (B, a, b) \longrightarrow (B', a', b') \longrightarrow 0$$

Thus the sheaf corresponding to (B, a, b) contains torsion.

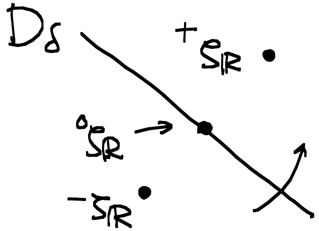
<u>Example</u> $\quad B_0 \leftrightarrow \mathcal{O}_{\mathbb{P}^1}(-n)$

for $\quad n\zeta^{(0)}_{\mathbb{R}} + (n+1)\zeta^{(1)}_{\mathbb{R}} = 0$



<u>Case 2°.</u> $\theta = \delta$ (level 0 hyperplane)

$$
\begin{tikzcd}
\mathfrak{M}_{+\zeta} \arrow[r, phantom, "\supset"] \arrow[drr] & {}^{+}\delta \arrow[r] & \mathfrak{M}_{0\zeta} \setminus \mathfrak{M}^{s}_{\zeta} \arrow[d, phantom, "\cap"] & {}^{-}\delta \arrow[l] \arrow[r, phantom, "\subset"] & \mathfrak{M}_{-\zeta} \arrow[dll] \\
 & & \mathfrak{M}_{0\zeta} & &
\end{tikzcd}
$$



a $\zeta_{\mathbb{R}}$-stable representation is either

or a) $\mathfrak{M}^{s}_{0\zeta}(V; W) \quad W \neq 0$

b) a point B in $\mathfrak{M}_{0\zeta}(\delta, 0) \cong X_{0\zeta}$

stratification of $\mathfrak{M}_{0_\zeta}(V,W)$

$$\mathfrak{M}_{0_\zeta}(V,W) \cong \coprod_{\substack{n=|\lambda| \\ V=V'+n\delta}} \mathfrak{M}^s_{0_\zeta}(V',W) \times S^n_\lambda X_{0_\zeta}$$

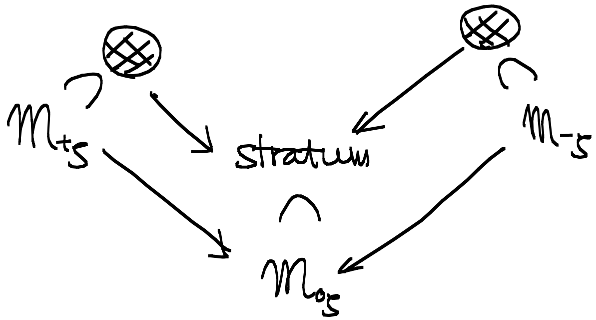
$\lambda=(\lambda_1 \geqq \lambda_2 \geqq \cdots)$ $\quad S^n_\lambda X_{0_\zeta} = \{ \sum \lambda_i x_i \mid x_i : \text{distinct points in } X_{0_\zeta} \}$

$\mathfrak{M}_{0_\zeta} \setminus \mathfrak{M}^s_{0_\zeta}$: union of stratum with $|\lambda| \neq 0$.

- different points have no extensions.
- But $\mathrm{Ext}^1(x_i, x_i) \cong T_{x_i} X_{0_\zeta}$: 2-dimensional

$\Longrightarrow$ difference from real root case.

Over each stratum, fibers of the projection $\mathfrak{M}_{-\zeta} \to \mathfrak{M}_{0\zeta}$ are



quot-scheme
$= \{ \mathcal{O}^{\oplus \mathrm{rank}} \twoheadrightarrow Q \mid \mathrm{Supp}\, Q = \sum \lambda_i x_i \}$

Opposite fibers are hard to describe ...... ("dual quot scheme")
But isomorphic to the quot scheme as in

$$Gr(m, N) \cong Gr(N-m, N) \quad \text{non-canonically.}$$

"Jordan" flop.

If we consider more general quiver varieties, and cross the wall defined by a root θ with
$(\theta, \theta) = 2 - 2g \qquad (g \geq 0)$,
we have "$(g+1)$-Jordan" flop.